Concurrence for general multipartite states



2006 J. Phys. A: Math. Gen. 39 15225

(http://iopscience.iop.org/0305-4470/39/49/010)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 15225–15229 doi:10.1088/0305-4470/39/49/010

# Concurrence for general multipartite states

**Hoshang Heydari**

Institute of Quantum Science, Nihon University, 1-8 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan

Received 4 October 2006, in final form 2 November 2006
Published 21 November 2006
Online at stacks.iop.org/JPhysA/39/15225

**Abstract**
We construct a generalized concurrence for general multipartite states based on local W-class and GHZ-class operators. We explicitly construct the corresponding concurrence for three-partite states. The construction of the concurrence is interesting since it is based on local operators and therefore gives some hints on the classification of multipartite states.

PACS number: 03.65.Ud

## 1. Introduction

Concurrence is one of the most applied measures of entanglement. In recent years there have been some proposals to generalize this measure of entanglement to general multipartite states [1, 2]. Recently, we have also defined concurrence classes for multi-qubit mixed states based on an orthogonal complement of a positive operator valued measure (POVM) on quantum phase [3]. Moreover, we have constructed different concurrence classes for general pure multipartite states in [4]. In this paper, we will construct generalized concurrence for pure general multipartite states based on the complement of a POVM on quantum phase. In particular, by rewriting orthogonal complement of a POVM on quantum phase as sums and taking the expectation value of each of these operators, we are able to construct a general formula for concurrence.

We will consider a general multipartite quantum system with $m$ subsystems which we denote as $\mathcal{Q} = \mathcal{Q}(N_1, N_2, \ldots, N_m)$, where $N_j$ are dimension of subsystem $j$ for all $1 \leqslant j \leqslant m$ and denoting its general state as $|\Phi\rangle = \sum_{l_1=1}^{N_1} \cdots \sum_{l_m=1}^{N_m} \alpha_{l_1,l_2,\ldots,l_m} |l_1, l_2, \ldots, l_m\rangle \in \mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the $j$ th Hilbert space is given by $N_j = \dim\left(\mathcal{H}_{\mathcal{Q}_j}\right)$. Note also that $\mathcal{Q}(N_1, N_2, \ldots, N_m)$ is just a notation for a $m$-partite quantum system and it is not a function. Moreover, let $\rho_{\mathcal{Q}} = \sum_{i=1}^{\mathrm{N}} p_i |\Phi_i\rangle\langle\Phi_i|$, for all $0 \leqslant p_i \leqslant 1$ and $\sum_{i=1}^{\mathrm{N}} p_i = 1$, denote a density operator acting on the Hilbert space $\mathcal{H}_{\mathcal{Q}}$. Finally, let us introduce a complex conjugation operator $\mathcal{C}_m$ that acts on a general multipartite state

  

$|\Phi\rangle$ as $\mathcal{C}_m|\Phi\rangle = \sum_{l_1=1}^{N_1} \cdots \sum_{l_m=1}^{N_m} \alpha^*_{l_1,l_2,\ldots,l_m}|l_1, l_2, \ldots, l_m\rangle$. The most well-known examples of multi-qubit states are $|\Psi_{W^m}\rangle$ and $|\Psi_{\mathrm{GHZ}^m}\rangle$ states. These quantum states are defined by

$$|\Psi_{W^m}\rangle = \frac{1}{\sqrt{m}}(|1, \ldots, 1, 2\rangle + |1, \ldots, 2, 1\rangle + \cdots + |2, 1, \ldots, 1\rangle) \tag{1}$$

and $|\Psi_{\mathrm{GHZ}^m}\rangle = \frac{1}{\sqrt{2}}(|1, \ldots, 1\rangle + |2, \ldots, 2\rangle)$. In the following section, we will call our local operators based on these classes of states.

## 2. General multipartite states

In this section, we will construct concurrence for general pure multipartite states $\mathcal{Q}^p(N_1, \ldots, N_m)$, where the superscript $p$ indicates that we are only considering pure multipartite states. In our construction, we will use linear operators that are constructed by the orthogonal complement of POVM on quantum phase [3, 4]. The POVM for each subsystem $\mathcal{Q}_j$ is defined by

$$\Delta_{\mathcal{Q}_j}(\varphi_{k_j,l_j}) = \sum_{l_j,k_j=1}^{N_j} \mathrm{e}^{\mathrm{i}\varphi_{k_j,l_j}}|k_j\rangle\langle l_j|, \tag{2}$$

where $\varphi_{k_j,l_j} = -\varphi_{l_j,k_j}\left(1 - \delta_{k_j l_j}\right)$. Moreover, the orthogonal complement of our POVM is given by $\widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi_{k_j,l_j}\right) = \mathcal{I}_{N_j} - \Delta_{\mathcal{Q}_j}\left(\varphi_{k_j,l_j}\right)$, where $\mathcal{I}_{N_j}$ is the $N_j$-by-$N_j$ identity matrix for subsystem $j$. For a $m$-partite quantum system we construct a operator (matrix) by taking the tensor product of $m$ subsystems as follows:

$$\widetilde{\Delta}_{\mathcal{Q}}\left(\varphi_{k_1,l_1}, \ldots, \varphi_{k_m,l_m}\right) = \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi_{k_1,l_1}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_m}\left(\varphi_{k_m,l_m}\right), \tag{3}$$

where $\widetilde{\Delta}_{\mathcal{Q}}\left(\varphi_{k_1,l_1}, \ldots, \varphi_{k_m,l_m}\right)$ has phases that are sums or differences of phases originating from two and $m$ subsystems. That is, in the latter case the phases of $\widetilde{\Delta}_{\mathcal{Q}}\left(\varphi_{k_1,l_1}, \ldots, \varphi_{k_m,l_m}\right)$ take the form $\left(\varphi_{k_1,l_1} \pm \varphi_{k_2,l_2} \pm \cdots \pm \varphi_{k_m,l_m}\right)$ and identification of these joint phases makes our distinguishing possible. Thus, we can define linear operators for the $\mathrm{W}^m$ class which are sums and differences of phases of two subsystems, i.e. $\left(\varphi_{k_{r_1},l_{r_1}} \pm \varphi_{k_{r_2},l_{r_2}}\right)$. That is, for the $\mathrm{W}^m$ class we have

$$\widetilde{\Delta}^{\mathrm{W}^m}_{\mathcal{Q}_{r_1 r_2}}(\Lambda_m) = \mathcal{I}_{N_1} \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\left(\varphi^{\frac{\pi}{2}}_{k_{r_1},l_{r_1}}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}\left(\varphi^{\frac{\pi}{2}}_{k_{r_2},l_{r_2}}\right) \otimes \cdots \otimes \mathcal{I}_{N_m}, \tag{4}$$

where $1 \leqslant r_1 < r_2 \leqslant m$ and the notation $\widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi^{\frac{\pi}{2}}_{k_j,l_j}\right)$ means that we evaluate $\widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi_{k_j,l_j}\right)$ at $\varphi_{k_j,l_j} = \pi/2$ for all $k_j, l_j$. In order to simplify our presentation, we have used $(\Lambda_m) = (k_1, l_1; \ldots; k_m, l_m)$ as an abstract multi-index notation. Next, we could write the linear operator $\widetilde{\Delta}^{\mathrm{W}^m}_{\mathcal{Q}_{r_1 r_2}}(\Lambda_m)$ as a direct sum of the upper and lower anti-diagonal

$$\widetilde{\Delta}^{\mathrm{W}^m}_{\mathcal{Q}_{r_1 r_2}}(\Lambda_m) = \mathfrak{U}\widetilde{\Delta}^{\mathrm{W}^m}_{\mathcal{Q}_{r_1 r_2}}(\Lambda_m) + \mathfrak{L}\widetilde{\Delta}^{\mathrm{W}^m}_{\mathcal{Q}_{r_1 r_2}}(\Lambda_m). \tag{5}$$

For the $\mathrm{GHZ}^m$ class, we define linear operators based on our POVM which are sums and differences of phases of $m$-subsystems, i.e. $\left(\varphi_{k_{r_1},l_{r_1}} \pm \varphi_{k_{r_2},l_{r_2}} \pm \cdots \pm \varphi_{k_m,l_m}\right)$. That is, for the $\mathrm{GHZ}^m$ class we have

$$\widetilde{\Delta}^{\mathrm{GHZ}^m}_{\mathcal{Q}_{r_1 r_2}}(\Lambda_m) = \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi^{\pi}_{k_1,l_1}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\left(\varphi^{\frac{\pi}{2}}_{k_{r_1},l_{r_1}}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}\left(\varphi^{\frac{\pi}{2}}_{k_{r_2},l_{r_2}}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_m}\left(\varphi^{\pi}_{k_m,l_m}\right), \tag{6}$$

where $\widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi^{\pi}_{k_j,l_j}\right)$ indicates that we evaluate $\widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi_{k_j,l_j}\right)$ at $\varphi_{k_j,l_j} = \pi$ for all $k_j, l_j$. Note also that, in this case, we get an operator which has the structure of the Pauli operator $\sigma_x$ embedded in a higher-dimensional Hilbert space and coincides with $\sigma_x$ for a single-qubit. There are

$\frac{m(m-1)}{2}$ linear operators for the $\mathrm{GHZ}^m$ class. Next, we write the linear operators for the $\mathrm{GHZ}^m$ class as

$$\widetilde{\Delta}^{\mathrm{GHZ}^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m) = \mathfrak{P}_1\widetilde{\Delta}^{\mathrm{GHZ}^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m) + \mathfrak{P}_2\widetilde{\Delta}^{\mathrm{GHZ}^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m) + \cdots, \tag{7}$$

where the operators $\mathfrak{P}_i\widetilde{\Delta}^{\mathrm{GHZ}^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m)$ are constructed by pairing of the anti-diagonal elements of the POVM with sums and differences of quantum phases. For higher dimensional quantum systems, it is difficult to write $\widetilde{\Delta}^{\mathrm{GHZ}^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m)$ in terms of $\mathfrak{P}_i\widetilde{\Delta}^{\mathrm{GHZ}^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m)$. However, we will give an explicit expression for general three-partite states in the next section. Moreover, we define the linear operators for the $\mathrm{GHZ}^{m-1}$ class of $m$-partite states based on our POVM which are sums and differences of phases of $m-1$-subsystems, i.e., $\left(\varphi_{k_{r_1},l_{r_1}} \pm \varphi_{k_{r_2},l_{r_2}} \pm \ldots \varphi_{k_{m-1},l_{m-1}}\right)$. That is, for the $\mathrm{GHZ}^{m-1}$ class we have

$$\begin{aligned}\widetilde{\Delta}^{\mathrm{GHZ}^{m-1}}_{\mathcal{Q}_{r_1r_2,r_3\cdots r_{m-1}}}(\Lambda_m) &= \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\left(\varphi^{\frac{\pi}{2}}_{k_{r_1},l_{r_1}}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}\left(\varphi^{\frac{\pi}{2}}_{k_{r_2},l_{r_2}}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_3}}\left(\varphi^{\pi}_{k_{r_3},l_{r_3}}\right)\\ &\quad \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_{m-1}}}\left(\varphi^{\pi}_{k_{r_{m-1}},l_{r_{m-1}}}\right) \otimes \mathcal{I}_{N_m},\end{aligned} \tag{8}$$

where $1 \leqslant r_1 < r_2 \leqslant r_3 \leqslant \cdots \leqslant r_{m-1} \leqslant m$. Note that we need to write these operators also as direct sums as we did for the $\mathrm{GHZ}^m$ class since they belong to the same operator class. Then, for a general pure state, let

$$\begin{aligned}\mathcal{C}\left(\mathcal{Q}^{W^m}_{r_1r_2}\right) &= \sum_{\forall k_j,l_j}\left(\left|\langle\Phi|\mathfrak{U}\widetilde{\Delta}^{W^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m)\mathcal{C}_m\Phi\rangle\right|^2 + \left|\langle\Phi|\mathfrak{L}\widetilde{\Delta}^{W^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m)\mathcal{C}_m\Phi\rangle\right|^2\right),\\ \mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^m}_{r_1r_2}\right) &= \sum_{\forall k_j,l_j}\sum_{i\geqslant m-2}\left|\langle\Phi|\mathfrak{P}_i\widetilde{\Delta}^{\mathrm{GHZ}^m}_{\mathcal{Q}_{r_1r_2}}(\Lambda_m)\mathcal{C}_m\Phi\rangle\right|^2\end{aligned} \tag{9}$$

and, e.g.,

$$\mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^{m-1}}_{r_1r_2,r_3\cdots r_{m-1}}\right) = \sum_{\forall k_j,l_j}\sum_{i\geqslant m-3}\left|\langle\Phi|\mathfrak{P}_i\widetilde{\Delta}^{\mathrm{GHZ}^{m-1}}_{\mathcal{Q}_{r_1r_2,r_3\cdots r_{m-1}}}(\Lambda_m)\mathcal{C}_m\Phi\rangle\right|^2. \tag{10}$$

Then the concurrence is defined by taking the square root of the summands as follows:

$$\begin{aligned}\mathcal{C}(\mathcal{Q}^p(N_1,\ldots,N_m)) &= \left(\mathcal{N}_m\left\{\sum_{1\leqslant r_1<r_2\leqslant m}\mathcal{C}\left(\mathcal{Q}^{W^m}_{r_1r_2}\right) + \sum_{1\leqslant r_1<r_2\leqslant m}\mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^m}_{r_1r_2}\right)\right.\right.\\ &\quad\left.\left. + \sum_{1\leqslant r_1<r_2\leqslant r_3\leqslant\cdots\leqslant r_{m-1}\leqslant m}\mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^{m-1}}_{r_1r_2,r_3\cdots r_{m-1}}\right) + \cdots\right\}\right)^{1/2},\end{aligned} \tag{11}$$

where $\mathcal{N}_m$ is a normalization constant. This concurrence vanishes on product state by definition. It also gives a reasonable measure of entanglement by construction. Note that for three-partite states our concurrence consists of two parts—$\mathcal{C}\left(\mathcal{Q}^{W^3}_{r_1r_2}\right)$ and $\mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^3}_{r_1r_2}\right)$—which we will discuss in the next section. However, for four-partite states we have $\mathcal{C}\left(\mathcal{Q}^{W^3}_{r_1r_2}\right)$, $\mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^3}_{r_1r_2}\right)$, and $\mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^3}_{r_1r_2,r_3}\right)$. Moreover, we can in principle define a concurrence for arbitrary multipartite states as

$$\mathcal{C}(\mathcal{Q}(N_1,\ldots,N_m)) = \inf_{\Phi}\sum_i p_i\mathcal{C}(|\Phi_i\rangle), \tag{12}$$

where infimum are taken over all pure decompositions of $\rho_Q$ and $\mathcal{C}(|\Phi_i\rangle)$ is given by $\mathcal{C}(\mathcal{Q}^p(N_1,\ldots,N_m))$ for all $i$. Our concurrence is constructed by a set of local operators which we have called $W^m$ and $\mathrm{GHZ}^m$ classes of operators. Thus, this construction of concurrence not only can quantify the entanglement of multipartite state but also it gives some hints on the possible classification of such quantum systems. As an example let us consider the multi-qubit $|\Psi_{W^m}\rangle$ state. Then, the concurrence measure for the $|\Psi_{W^m}\rangle$ state gives

$\mathcal{C}^p(\mathcal{Q}(2,\ldots,2)) = \left(\frac{2(m-1)}{m}\mathcal{N}_m\right)^{1/2}$. The normalization constant $\mathcal{N}_m$ can be chosen in such a way that $0 \leqslant \mathcal{C}(\mathcal{Q}(N_1,\ldots,N_m)) \leqslant 1$.

## 3. General pure three-partite states

In this section, we will construct concurrence for a general pure three-partite quantum system $\mathcal{Q}^p(N_1, N_2, N_3)$ based on the orthogonal complement of our POVM. For three-partite states, we have two different joint phases in our POVM, those which are sums and differences of phases of two subsystems, i.e. $\left(\varphi_{k_1,l_1} \pm \varphi_{k_2,l_2}\right)$ and those which are the sums and differences of phases of three subsystems, i.e. $\left(\varphi_{k_1,l_1} \pm \varphi_{k_2,l_2} \pm \varphi_{k_3,l_3}\right)$. The first one identifies the $\mathrm{W}^3$ class operator and the second one identifies the $\mathrm{GHZ}^3$ class operator. For the $\mathrm{W}^3$ class, we have

$$\sum_{1\leqslant r_1<r_2\leqslant 3} \mathcal{C}\left(\mathcal{Q}^{W^3}_{r_1r_2}\right) = \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \sum_{k_3=l_3=1}^{N_3} \left|\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\right|^2 + \sum_{l_1>k_1=1}^{N_1} \sum_{l_3>k_3=1}^{N_3} \sum_{k_2=l_2=1}^{N_2} \left|\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\right|^2 + \sum_{l_2>k_2=1}^{N_2} \sum_{l_3>k_3=1}^{N_3} \sum_{k_1=l_1=1}^{N_1} \left|\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\right|^2, \tag{13}$$

and for the $\mathrm{GHZ}^3$ class, we have

$$\sum_{1\leqslant r_1<r_2\leqslant 3} \mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^3}_{r_1r_2}\right) = \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \sum_{l_3>k_3=1}^{N_3} \Big[\left|\alpha_{k_1,l_2,l_3}\alpha_{l_1,k_2,k_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\right|^2 + \left|\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3} - \alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3}\right|^2 + \left|\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3} - \alpha_{k_1,l_2,l_3}\alpha_{l_1,k_2,k_3}\right|^2 + \left|\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\right|^2 + \left|\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3} - \alpha_{k_1,l_2,l_3}\alpha_{l_1,k_2,k_3}\right|^2 + \left|\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\right|^2\Big]. \tag{14}$$

Note that these expressions are not equal to our W class and GHZ class concurrences constructed in [4], where we have constructed our concurrences classes based on the direct use of two classes of operators. Thus the concurrence for a general pure three-partite state is given by

$$\mathcal{C}(\mathcal{Q}^p(N_1, N_2, N_3)) = \left(\mathcal{N}_3\left[\sum_{1\leqslant r_1<r_2\leqslant 3} \mathcal{C}\left(\mathcal{Q}^{W^3}_{r_1r_2}\right) + \sum_{1\leqslant r_1<r_2\leqslant 3} \mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^3}_{r_1r_2}\right)\right]\right)^{1/2}. \tag{15}$$

This concurrence also coincides with the generalized concurrence for three-partite states [1]. Moreover, for $m$-partite states with $m \geqslant 3$, our concurrence is not equal to concurrence tensor [5].

## Acknowledgments

The author acknowledges useful comments from Jonas Söderholm. The author gratefully acknowledges the financial support of the Japan Society for the Promotion of Science (JSPS).

## References

[1] Albeverio S and Fei S M 2001 *J. Opt. B: Quantum Semiclass. Opt.* **3** 223
[2] Mintert F, Kus M and Buchleitner A 2005 *Phys. Rev. Lett.* **95** 260502
[3] Heydari H 2005 *J. Phys. A: Math. Gen.* **38** 11007–17
[4] Heydari H 2005 *J. Phys. A: Math. Gen.* **38** 8667–79
[5] Heydari H and Björk G 2005 *Quantum Inform. Comput.* **5** 146–55